B.sc(H) part2 paper 3
Topic:Theorem (homomorphism & isomorphism)
Subject:mathematics
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Theorem 1. **To show that the relation '$\cong$' of being isomorphic is an equivalence relation on any set S of groups.**

Proof : We shall prove that the relation of isomorphism denoted by $\cong$ in the set S of all groups is reflexive, symmetric and transitive. Let $G, H, K \in S$.

Relexive : $G \cong G$,

Let f be the identity mapping on G i.e. $f : G \to G$

such that $f(x) = x$ for all $x \in G$.

Obviously f is one-one onto.

Also, Let $x, y \in G$, then $f(x) = x$ and $f(y) = y$.

$\therefore \quad f(xy) = xy$

$\qquad\qquad = f(x) f(y)$.

Hence f preserves operations in G and G. Thus f is an isomorphism of G onto G. Hence $G \cong G$.

Symmetric : i.e. $G \cong H \Rightarrow H \cong G$.

Let $G \cong H$. Let f be an isomorphism of G onto H. Then f is one-one onto and preserves operations in G and H.

Since f is one-one onto, therefore it is invertible, i.e. f^{-1} exists. Also we know that the inverse function f^{-1} is one-one onto.

Now we shall show that $f^{-1} : H \to G$ also preserves operation.

Let $x', y' \in H$. Then there exist elements $x, y \in G$ such that $f^{-1}(x') = x$ and $f^{-1}(y') = y$

$$\Rightarrow f(x) = x', f(y) = y' \quad \ldots(1)$$

Now, $f^{-1}(x' y') = f^{-1}[f(x) f(y)]$; from (1)

$= f^{-1}[f(xy)]$; since $f(xy) = f(x) f(y)$

$= xy$; from definition of f^{-1}

$= f^{-1}(x') f^{-1}(y')$ from (1)

$\therefore$ f^{-1} preserves operation in H and G.

Hence $H \cong G$.

Transitive : i.e. $G \cong H,\ H \cong K \Rightarrow G \cong K$.

Suppose G is isomorphic to H and H is isomorphic to K.

Further suppose that $f : G \to H$ and $g : H \to K$ are the respective isomorphic mappings.

Then $g o f : G \to K$.

If both f and g are one -one onto, we know that the composite mapping

$g o f : G \to K$ defined by

$g of(x) = g[f(x)]$ for all $x \in G$

is also one-one onto.

Further, if $x, y \in G$, then

$(g o f)(xy) = g[f(xy)]$

$= g[f(x)f(y)]$, $\because$ f is an isomorphism

$= g[f(x)g[f(y)]$; g is an isomorphism

$= [(g o f)(x)] [(g o f)(y)]$

Hence $g o f$ preserves operations in G and K.

$\therefore$ $g o f$ is an isomorphism of G on K and $\therefore G \cong K$.

Hence the relation of isomorphism in the set of groups is an equivalence relation.

Theorem 2

Let $f : G \to G'$ be a homomorphism of groups.

(i) If e and e' be the identities in G and G' respectively then $f(e) = e'$.

(ii) If $f(a) = a'$, then $f(a^{-1}) = (a')^{-1}$.

i.e. $f(a^{-1}) = [f(a)]^{-1}$ for all $a \in G$

In other words, if $f : G \to G'$ be a homomorphism, then their identities correspond and their inverses correspond.

(iii) If the order of $a \in G$ is finite, then the order of $f(a)$ is a divisor of the order of a.

Proof : (i) Let $f(e) = e'$ where e is the identity of G and $e' \in G'$.

If f is a homomorphism, we have to prove that e' is the identity of G'.

Take $\quad x \in G$ and let $f(x) = x'$ $(x' \in G')$.

Now $\quad x = ex,$

$\therefore \quad f(x) = f(ex)$

$= f(e) \cdot f(x)$; since f is a homomorphism

$\Rightarrow \quad x' = e'x'$

which means that e' (i.e. $f(e)$) is the identity in G'.

(ii) Given $f(a) = a'$.

Now $\quad aa^{-1} = e$ (the identity in G)

$\therefore \quad f(aa^{-1}) = f(e) = e'$; from (i)

That is, $\quad f(a) \cdot f(a^{-1}) = e'$ since f is a homomorphism

i.e. $\quad a' f(a^{-1}) = e'$

which means that the inverse of a' is $f(a^{-1})$.

That is, $\quad f(a^{-1}) = (a')^{-1} = [f(a)]^{-1}$.

(iii) Let $a \in G$ and $o(a) = m$.

Thus, we have $o(a) = m \Rightarrow a^m = e$.

$\therefore \quad f(a^m) = f(e)$

$\Rightarrow f(aaa \ldots \text{to } m \text{ factors}) = e'$

$\Rightarrow f(a)\, f(a) \ldots m \text{ times} = e' \Rightarrow [f(a)]^m = e'$.

Hence if n is the order of $f(a)$ in G', then n must be a divisor of m; i.e. $o(f(a))$ is a divisor of $o(a)$.

Theorem : Show that every isomorphic image of a cyclic group is again cyclic.

Proof : Let $G = \langle a \rangle$ be a cyclic group generated by a. Let G' be an isomorphic image of G under the isomorphism f i.e. $f : G \to G'$.

The elements of G' are the images of the elements of G under the mapping f.

Let $f(a^n) \in G'$ be the image of the element $a^n \in G$.

We have,

$f(a^n) = f(a\, a\, a \ldots \text{to } n \text{ factors})$

$= f(a)\, f(a)\, f(a) \ldots$ to n factors, since f is an isomorphism.

$= [f(a)]^n$

Thus we see that every element of G' can be expressed as an integral power of $f(a)$.

Hence G' is cyclic and $f(a)$ is a generator of G'.

Theorem : Show that every homomorphic image of an Abelian group is Abelian.

Soln. : Let G be an Abelian group. Let f be a homomorphic mapping of G onto G'. Then G' is a homomorphic image of G.

It is to prove that G' is Abelian.

Let a', b' be any two elements of G'.

Then $f(a) = a'$ and $f(b) = b'$ for some $a, b \in G$.

We have, $a'b' = f(a)\, f(b) = f(ab)$

$\because f$ is homomorphic mapping

$= f(ba)$; $\because$ G is Abelian

$= f(b)\, f(a) = b'a'$

Hence G' is Abelian.